

Quantum discord is Bohr's notion of non-mechanical disturbance introduced in his answer to EPR

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Abstract

By rigorously formalizing the Einstein-Podolsky-Rosen (EPR) argument, and Bohr's reply, one can appreciate that both arguments were technically correct. Their opposed conclusions about the completeness of quantum mechanics hinged upon an explicit difference in their criteria for when a measurement on Alice's system can be regarded as not disturbing Bob's system. The EPR criteria allow their conclusion (incompleteness) to be reached by establishing the physical reality of just a single observable q (not a conjugate pair q and p), but I show that Bohr's definition of disturbance prevents the EPR chain of reasoning from establishing even this. Moreover, I show that Bohr's definition is intimately related to the asymmetric concept of quantum discord from quantum information theory: if and only if the joint state has no Alice-discord, she can measure any observable without disturbing (in Bohr's sense) Bob's system. Discord can be present even when systems are unentangled, and this has implications for our understanding of the historical development of notions of quantum nonlocality.

1. Introduction

No paper is more seminal for the fields of quantum foundations and quantum information than the 1935 Einstein, Podolsky and Rosen (EPR) paper [1]. It was the first to discuss the correlations between the measurement results of two parties who share an entangled state. Indeed, the term *entangled* was introduced by Schrödinger in the same year [2], in a paper motivated (as Schrödinger explicitly notes) by the EPR paper. In another paper of that year [3], Schrödinger introduced the term 'steering' to describe the EPR-phenomenon, a notion which has recently been formalized in quantum informational terms [4, 5], and shown to be a sort of quantum nonlocality intermediate between entanglement and Bell-nonlocality. This last notion was of course introduced in Bell's 1964 paper [6]. Although it sunk Einstein's dream of a local deterministic theory of quantum phenomena, it too was directly inspired by the EPR paper, being entitled "On the Einstein Podolsky Rosen Paradox".

In this paper, I show that yet another famous paper written in reaction to the EPR paper introduced a concept of quantum nonlocality which has only relatively recently

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Preprint submitted to Elsevier

August 27, 2012

been formalized (from an independent perspective [7, 8]) and which is of considerable current interest [9–11]. The paper is Bohr’s 1935 response to EPR [12], and the concept is *quantum discord*, introduced in 2001 [7, 8]. I am using the term ‘quantum nonlocality’ here (and above) in a very broad sense: the phenomenon requires two (or more) parties who could be arbitrarily separated in space and who must share a quantum state. Indeed, the term has been used (‘quantum nonlocality without entanglement’ [13]) for an effect that is intimately related to quantum discord.

Bohr famously gave his paper the same title as the EPR paper: “Can quantum mechanical description of physical reality be considered complete?” But of course while EPR answered this questions ‘no’, Bohr answered ‘yes’. There are different opinions [14–17] about whether EPR, or Bohr, made logical errors in this debate, or whether (as I advocate here) the disagreement can be understood as arising from EPR and Bohr using different definitions. Under the latter scenario, their different definitions presumably reflect their different conceptions of the nature of the world and our relation to it, and one could still discuss whether Bohr’s or Einstein’s world view is to be preferred. It is not the purpose of this paper to delve into such philosophical matters, but rather to isolate the essence of the disagreement and to formalize it. As I will show, the most reasonable formalization of the arguments of EPR and Bohr allow both of them to be logically correct, with an *explicit* difference in one definition (that of *disturbance*) being responsible for their different conclusions.

Quantum discord δ was defined by Ollivier and Zurek in 2001, as a “measure of the quantumness of correlations” between two parties, Alice and Bob, who share a quantum state [8]. Roughly, the Alice-Discord $\delta(\beta|a)$ quantifies the amount of information Alice has about Bob’s system β which is lost when Alice makes a projective measurement a on her system. In the same year, Henderson and Vedral considered a similar quantity, without naming it, but required that it be *minimized* over all possible (including non-projective) measurements by Alice [7]. This is now usually taken as the definition of the discord of a state [9–11]. I show that the relation of discord to Bohr’s 1935 paper is the following. If and only if the discord is zero, Alice can measure any observable on her system in a way that does not disturb (in Bohr’s sense) Bob’s system.

Because this paper uses a relatively large number of formal concepts, and employs symbolic logical in some parts, it is helpful to introduce abbreviations for many of these concepts. This is done in Sec. 2, which also contains some preliminary definitions used later in the paper. In Sec. 3 I formalize the EPR argument, and also give a simplified version which involves only a single observable q_β for Bob’s system. Then I formalize Bohr’s counter-argument in Sec. 4 and show that it works against both the original EPR argument and the simplified version. In Sec. 5 I show the relation between the EPR argument and entanglement, while in Sec. 6 I show how Bohr’s notion of disturbance relates to the notion of quantum discord. Section 7 concludes with a discussion of the implications of this work, particularly in the light of Bell’s theorem.

2. Notation and some definitions

2.1. Preliminary notation

Consider an experiment involving two parties, or agents, Alice and Bob. They can freely choose between various actions, which can be regarded as controlling their measurement settings. Following Bell’s notation [6], I will denote these a and b respectively.

These settings then yield outcomes A and B respectively (Bell's notation). The *preparation* c (Bell's notation) is the macroscopic event (or series of events) initiating the experiment — apart from the measurement settings controlled by Alice and Bob — the details of which remain fixed between one experimental run and the next. For c to have any role in the *correlations* between A and B , it (or parts of it) must lie in the past light cones of both A and B .

To connect to the language of EPR and Bohr, it is necessary to talk of *systems*. To give this an operational meaning (that is, one grounded in macroscopic events), assume that events a and A are locatable within a space time region α which is disjoint from another region β containing b and B . We can then identify Alice's and Bob's systems with α and β respectively. We assume that c lies in the backward light cones of both α and β , which I denote $t(c) < t(\alpha), t(\beta)$. For Bell nonlocality, it is often assumed that α is space-like separated from β , so that it is true neither that $t(\alpha) < t(\beta)$ nor that $t(\alpha) > t(\beta)$. This arrangement was never assumed by EPR or Bohr, but is certainly compatible with the physics they considered. EPR and Bohr talk about one party (say Alice) making *predictions* regarding Bob's system, so it does seem reasonable to require that $t(\alpha) > t(\beta)$ *not* be true.

EPR also refer to *physical quantities* pertaining to systems α and β . I will denote these by $\hat{a} \in \mathbb{P}_\alpha$ and $\hat{b} \in \mathbb{P}_\beta$. I use a hat because in quantum mechanics physical quantities are represented by operators, but there is no implication here that quantum mechanics must be correct. Because there are many ways to measure a physical quantity, \hat{a} is associated with an equivalence class of settings: $a \in \mathbb{S}_{\hat{a}}$, and similarly for \hat{b} and b . This equivalence class will be defined formally below. Finally, I also define the set of possible outcomes A for a measurement setting $a \in \mathbb{S}_{\hat{a}}$ as $\mathbb{O}_{\hat{a}}$ and similarly for B . In the remainder of this paper, unless otherwise specified, the symbol $\forall a$ is to be understood as meaning all possible a ; that is, $\forall a \in \mathbb{S}_{\hat{a}}$ for some $\hat{a} \in \mathbb{P}_\alpha$; and likewise for b , A , and B , *mutatis mutandis*.

2.2. Some definitions

I can now present some basic definitions using the above events and variables.

Definition 1 (Phenomenon). A phenomenon $\phi \equiv (c, \mathbb{P}_\alpha^\phi, \mathbb{P}_\beta^\phi)$ is defined by the complete set of relative frequencies for the outcomes A and B :

$$\{f_\phi(A, B|a, b, c) : A, B, a, b\}$$

for some preparation c and for all physical quantities $\hat{a} \in \mathbb{P}_\alpha^\phi \subseteq \mathbb{P}_\alpha$ and $\hat{b} \in \mathbb{P}_\beta^\phi \subseteq \mathbb{P}_\beta$.

Here I am using the notation $\{f : g\}$ to mean the set of values of f , where g is the set-index, which ranges over all allowed values of g . I use the term 'relative frequency' rather than 'probability' to emphasize that a phenomenon is something empirically observed. Of course it may also be theoretically predicted, and in particular I will use the term *quantum phenomenon* for a phenomenon that is predicted by quantum mechanics. The set of quantum phenomena will be denoted \mathbb{Q} . An *operational theory* is a means of calculating all the relative frequencies $f_\phi(A, B|a, b, c)$ pertaining to given phenomenon $\phi \equiv (c, \mathbb{P}_\alpha^\phi, \mathbb{P}_\beta^\phi)$, and which postulates no variables distinct from the specified macroscopic

events. The notion of equivalence class of measurement settings is also operational, and may depend upon the preparation c :

$$a \in \mathbb{S}_{\hat{a}|c} \iff \forall a' \in \mathbb{S}_{\hat{a}|c}, A, B, b, \left[f_\phi(A, B|a, b, c) = \sum_{A'} f_\phi(A', B|a', b, c) \delta_{A', A} \right] \quad (1)$$

Orthodox quantum theory (OQT) is the example *par excellence* of an operational theory, at least for preparation procedure c corresponding to a pure state and *sharp* measurements a and b (i.e. with rank-one probability operators [18]). This case is special because it has a unique set of relative frequencies for all observers, as there are no extra macroscopic variables that could be known to some observers but not others.

Next, we need to note that EPR use the terms *description of physical reality* and *theory* interchangeably. That is, they did not use the term *theory* to mean operational theory. Rather, they imagined that a *theory* could involve additional (*hidden*) variables, with postulated relations between the variables that could not be derived from the preparation c or observable relative frequencies alone. For the purpose of the debate between EPR and Bohr, we need consider only a single phenomenon ϕ (as defined by me above), and hence the class Θ_ϕ of theories describing this phenomenon. Thus,

Definition 2 (Theory). A theory $\theta \in \Theta_\phi$ for a phenomenon ϕ consists of

1. The set Λ_c^θ of values of λ .
2. A (non-negative) probability measure $d\mu_c^\theta(\lambda)$ on Λ_c^θ .
3. The (non-negative) conditional probabilities $P_\theta(A, B|a, b, c, \lambda)$

which reproduces the phenomenon for all $\hat{a} \in \mathbb{P}_\alpha^\phi, \hat{b} \in \mathbb{P}_\beta^\phi$:

$$\forall A, B, a, b, \int_{\Lambda_c^\theta} d\mu_c^\theta(\lambda) P_\theta(A, B|a, b, c, \lambda) = f_\phi(A, B|a, b, c)$$

EPR's intuition was that the hidden variables λ could allow a better *description of physical reality* than that given by an operational theory.

3. The EPR paper

The EPR paper grew out of decades of disquiet Einstein felt about quantum theory, exacerbated by the rise of the Copenhagen interpretation as the orthodox one [16]. In this paper I am not concerned with putting the EPR paper in the context of Einstein's earlier or later writings (unlike in [19]), but rather with presenting the formal argument in the EPR paper itself quite rigorously (unlike in [19]). The reason, as I explained in the Introduction, is that it was at the level of formal argument that Bohr responded to the EPR paper, and it is at this level that the connection to discord lies.

3.1. The EPR criteria

The EPR paper is of course concerned with completeness and, as with almost all concepts they introduce, they explain it reasonably precisely:

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the theory.*

Here, and — except where noted — below, the italics are as in the original. This criterion can be expressed using formal logic as

Criterion 3 (Completeness).

$$\forall \theta \in \Theta_\phi, \text{Com}(\theta) \implies \left[\forall \hat{b} \in \mathbb{P}_\beta^\phi, \mathcal{EPR}(\hat{b}|c) \implies \mathcal{Rep}_\theta(\hat{b}|c) \right].$$

Here I have used $\mathcal{Rep}_\theta(\hat{b}|c)$, standing for ‘is represented’, to denote EPR’s concept that \hat{b} ‘has a counterpart’ in theory θ , and $\mathcal{EPR}(\hat{b}|c)$ to mean that the property \hat{b} is an element of physical reality. The notation used in the above requires some comment. First, I use the notation $\mathcal{EPR}(\hat{b}|c)$, rather than $\mathcal{EPR}_\theta(\hat{b}|c)$ because EPR are quite clear that their sufficient condition (see below) for something to be an element of physical reality must come from the phenomenon, not the theory:

The elements of physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements.

Second, the notation $\mathcal{EPR}(\hat{b}|c)$, rather than $\mathcal{EPR}(\hat{b}|a, c)$ should not be read to imply that the reality of a physical property of system β is necessarily independent of the process of measurements performed upon system α . However, as I will discuss below, the sufficient criterion EPR use for $\mathcal{EPR}(\hat{b}|c)$ applies only in the case where the measurement a does not disturb the system β . In this case, they are again clear that

No reasonable definition of reality could be expected to permit [that] the reality of [properties of the second system] depend upon the process of measurements carried out on the first system, which does not disturb the second system in any way.

Thus in the case of interest, it must be the case that $\mathcal{EPR}(\hat{b}|c) = \mathcal{EPR}(\hat{b}|a, c)$. Third, EPR’s sufficient criterion (below) implies that it would be redundant to consider $\mathcal{EPR}(\hat{b}|b, c)$. It would not, however, be sensible to consider $\mathcal{EPR}(\hat{b}|B, b, c)$, as this would make EPR’s requirement of being able to predict the outcome (B) of the measurement completely trivial. Since we can thus restrict to considering $\mathcal{EPR}(\hat{b}|c)$, it follows that we must restrict the conditionals to what ‘has a counterpart’ in theory θ in the same way. Thus we obtain a local concept of representation, $\mathcal{Rep}_\theta(\hat{b}|c)$, and thus a concept of local elements of physical reality, which is critical to EPR’s argument [20].

EPR do not actually define what it means for \hat{b} to ‘have a counterpart in the theory’, but I think it is uncontroversial to take it to imply that the outcome of any measurement of \hat{b} is determined in the theory:

Criterion 4 (Representation in the theory).

$$\mathcal{Rep}_\theta(\hat{b}|c) \implies \forall B, b, \lambda, P_\theta(B|b, \lambda, c) \in \{0, 1\}.$$

Regarding elements of physical reality, EPR stress that “[a] comprehensive definition of reality is ... unnecessary for our purpose.” Instead, they give the following sufficient criterion:

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

Formally, we have:

Criterion 5 (Element of Physical Reality).

$$\mathcal{EPR}(\hat{b}|c) \iff \exists a : [\neg \text{Dist}(\beta|a, c) \wedge \text{Pre}(\hat{b}|a, c)].$$

Here $\text{Dist}(\beta|a, c)$ means that the measurement a disturbs the system β , while $\text{Pre}(\hat{b}|a, c)$ means that the measurement a makes the value of \hat{b} predictable. Like representation, predictability¹ can be defined uncontroversially:

Definition 6 (Predictability).

$$\text{Pre}(\hat{b}|a, c) \iff \forall B, b, A, f(B|A, a, b, c) \in \{0, 1\}.$$

Note the use here of the operationally determined relative frequencies f , not the theory probabilities P_θ . The second new notion — disturbance — requires clarification. EPR say (with my emphasis):

A definite value of the coordinate $[x]$, for a particle in the state given by Eq. (2) $[\psi(x) = e^{(2\pi i/\hbar)p_0 x}]$, is thus not predictable, but may be obtained only by direct measurement. Such a measurement however *disturbs* the particle and thus alters its state. After the coordinate is determined, the particle will no longer be in the state given by Eq. (2).

Here by ‘state’ EPR obviously mean the particle’s quantum state — its state according to OQT². We can avoid referring to quantum theory (as has been the case with my definitions so far) by recognizing that the state of system β in OQT can be identified with a list of the possible relative frequencies of results B that will be obtained upon

¹Note that it is important to distinguish between predictability and determinism; see e.g. Ref. [21].

²On the basis of other passages in the EPR paper, a case can be made that EPR’s notion of disturbance was meant to apply to the ‘real situation’ (to use one of Einstein’s later phrases) of a system, not just its quantum state [R. Spekkens, private communication (2012)]. One could formalize this by introducing a theory-dependence in the criterion for non-disturbance [i.e. $\neg \text{Dist}_\theta(\beta|a, c)$] by using the theoretical probabilities P_θ rather than the relative frequencies f . However one could not simply substitute this for $\text{Dist}(\beta|a, c)$ in Criterion 5 because that criterion uses, besides the disturbance which is in question, only operational concepts, namely $\mathcal{EPR}(\hat{b}|c)$ and $\text{Pre}(\hat{b}|a, c)$. However one could replace $\neg \text{Dist}(\beta|a, c)$ in Criterion 5 by $\exists \theta : \neg \text{Dist}_\theta(\beta|a, c)$, since every theory must reproduce the operational predictions, so if there is no disturbance in some theory then there will be no disturbance in the operational sense (the criterion I have advocated). Making this change would make no essential change to my interpretation of EPR and Bohr because EPR do not justify the lack of disturbance from examining the lack of change in either the relative frequencies, or the theoretical probabilities. Rather, they justify it from the fact that “the two systems no longer interact,” as discussed in the main text.

performing measurements b . For a measurement a to ‘alter the state’ means that at least one relative frequency is affected by choosing a rather than some other measurement a' (which could be the null measurement). Since EPR imply that the alteration of the state follows from the disturbance of the system (the ‘particle’), it seems reasonable to formalize their notion of non-disturbance as the sufficient criterion

Criterion 7 (No Disturbance à la EPR).

$$\neg \text{Dist}^{\text{EPR}}(\beta|a, c) \Leftarrow \forall a', B, b, f(B|a, b, c) = f(B|a', b, c).$$

Note the use of the EPR superscript here because later I will consider a different definition for disturbance, that proposed by Bohr. If α and β are space-like separate, then no-disturbance follows from Criterion 7 and the no-superluminal-signalling (from Alice to Bob) assumption. EPR themselves make the stronger assumption, that “since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.” Presuming that ‘no real change’ must mean no disturbance (see again the preceding footnote), it follows that EPR embraced the following

Assumption 8. *Presuming ‘ α and β no longer interact’ $\implies \forall a, \neg \text{Dist}^{\text{EPR}}(\beta|a, c)$.*

Thus if one disagreed with the operational Criterion 7 I have abstracted from the EPR paper, one could replace it, without significantly affecting any of my analysis or conclusions, with Assumption 8.

3.2. The EPR theorem

Having suitably formalized EPR’s definitions and criteria, we can now turn to what they prove. It will be apparent that the two EPR criteria, for completeness and for elements of physical reality, being necessary and sufficient respectively, are the minimum required for proving the EPR theorem. By contrast, the logical argument EPR employ, while correct, is unnecessary complicated. This must have contributed to Einstein’s complaint that the paper, apparently written for publication by Podolsky, was ‘smothered in formalism’ [22]³. Here I present a simpler, but logically equivalent, argument.

Consider the quantum phenomenon ϕ in which c is the preparation of a two-particle entangled state, such that at some time t_1 , $\hat{q}_\alpha = \hat{q}_\beta - x_0$, and $\hat{p}_\alpha = -\hat{p}_\beta$. Here the subscript α (β) denotes the particle present in Alice’s (Bob’s) lab at the time of measurement, which I take to be t_1 in both cases. Following EPR, take the particles to be ‘no longer interacting’, which will be the case if Alice’s and Bob’s labs are spatially separate. Under these conditions it follows that (i) $\text{Pre}(\hat{q}_\beta|q_\alpha, c)$, because if Alice measures q_α with result Q_α , she can predict that if Bob were to measure q_β he would find result $Q_\beta = Q_\alpha + x_0$. It is also the case that (ii) $\neg \text{Dist}^{\text{EPR}}(\beta|\hat{q}_\alpha, c)$, because there is no signalling from Alice to Bob. From Criterion 5 it follows that (iii) $\mathcal{EPR}(\hat{q}_\beta|c)$. By an identical argument, it follows that (iv) $\mathcal{EPR}(\hat{p}_\beta|c)$.

³The formal definitions and criteria I have used may also have been unpalatable to Einstein, but I think are necessary to best appreciate Bohr’s argument. After all, Bohr wrote in response to the EPR paper, not to any other, less formal but arguably clearer, statements by Einstein.

Now assume that (v) OQT, is complete. Here I regard OQT, for a particular phenomenon ϕ , as actually a set $O_\phi \subset \Theta_\phi$ of theories each which uses only the concepts of OQT (quantum states and operations). There might be some argument as to whether EPR's notion of completeness should be expected to hold when dealing with mixed states, but in the present situation the EPR correlations defined above can pertain only for a pure state, and indeed there is only one theory $\theta \in O_\phi$ which yields these correlations. Thus a defender of the completeness of O_ϕ must take this θ to be complete. Then, from Criterion 3, and results (iii) and (iv) above, it follows that (vi) for this θ , $\mathcal{Rep}_\theta(\hat{q}_\beta|c)$ and $\mathcal{Rep}_\theta(\hat{p}_\beta|c)$. By Criterion 4, this means that (vii) the theory θ determines both the value of \hat{q}_β and the value of \hat{p}_β . However, (vii) is obviously false because in OQT a pair of non-commuting operators do not have determined values under any circumstances. Thus the assumption of completeness (v) is false:

Theorem 9 (EPR, 1935).

$$\exists \phi \in Q : \forall \theta \in O_\phi, \neg \text{Com}(\theta).$$

Here Q is the set of all quantum phenomena.

As has been noted many times before — e.g. Refs. [16, 22, 23] — the EPR argument can be made even simpler. Given the EPR correlations, and hence the unique OQT θ which reproduces them, we do not need to consider a pair of observables at all. It is enough to assume completeness, and derive $\mathcal{Rep}_\theta(\hat{q}_\beta|c)$, for example. To obtain a contradiction, simply note that, in the EPR state, \hat{q}_β does not have a determined value (it in fact has an infinite uncertainty). If one does not assume that QM is correct, then there is a motivation not to discard the non-commuting variables version [19], but for the purposes of this paper the one-variable argument is just as good. Indeed, as we will see in Sec. 4.2, the simpler argument helps to highlight the generality of Bohr's counter-argument.

4. Bohr's Response

4.1. Formalizing Bohr's counter-argument

Bohr's response [12] to EPR's incompleteness theorem is notoriously hard to follow [15, 19, 24]. In the greater part of his reply, Bohr did not address the EPR entangled state, but rather discussed the complementarity of properties of a single particle, as he had previously done to defend the *consistency* of OQT against Einstein's earlier attacks. This defence was based upon the mechanical disturbance of the quantum system by the apparatus, which was not obviously relevant to the EPR scenario since the incompleteness lies in the quantum description of Bob's system, which need not interact with any apparatus. Bohr himself, when reviewing the Einstein–Bohr debates more than a decade later [25], contrasted the ‘lucidity’ of the EPR paper ⁴ with his own “inefficiency of expression which must have made it very difficult to appreciate the trend of the argumentation.”

⁴It must however be borne in mind that Bohr was writing this summary for a volume in honour of Einstein's 70th birthday.

Despite this, I now believe (contrary to [19]) that Bohr’s counter-argument can be interpreted as a precise rejection of one of EPR’s criteria, and that Bohr guides the reader to this in his later review. There, Bohr again refers back to his earlier defences of complementarity, summing up [25]: “As repeatedly stressed, the principal point is here that such measurements [of complementary properties] demand mutually exclusive experimental arrangements.” He then goes on to quote from his 1935 paper (truncated here by me):

From our point of view we now see that the wording of the above-mentioned criterion of physical reality proposed by Einstein, Podolsky and Rosen contains an ambiguity as regards the meaning of the expression ‘without in any way disturbing a system.’ Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system*. Since these conditions constitute an inherent element of the description of any phenomenon to which the term physical reality can be properly attached, we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete.

With the word ‘ambiguity’ here, Bohr heralds that there are two possible ways way to understand ‘disturbing a system’. In the short note in Nature [26] announcing his forthcoming reply to EPR, in the first sentence of the abstract of that reply [12], and in its introduction, Bohr identifies the supposed weakness of EPR’s argument as being an ‘essential ambiguity’ in their criterion for physical reality. This again shows that the ‘ambiguity’ passage is key to understanding his rebuttal of EPR.

Bohr seems, quite reasonably, to associate the EPR notion of disturbance, Criterion 7 (or Assumption 8), with ‘mechanical disturbance’ and concedes that there is certainly no disturbance of that sort on Bob’s system in the EPR scenario. But implicitly he rejects EPR’s disturbance definition as too narrow, and proposes a broader notion: a system β is disturbed by a measurement a if that measurement influences the “conditions which define the possible types of predictions regarding the future behaviour of the system.” What is it that defines these possible predictions by Alice regarding Bob’s system β ? Nothing but the joint quantum state, modulo any reversible transformations on system α . In operational (non specifically quantum) terms, a_1 is non-disturbing if and only if choosing to perform a_1 — rather than a' , say — does not preclude the possibility of obtaining, via some *subsequent* measurement a_2 , the same information about β as would have been yielded by the measurement a' : Of course we also want to avoid mechanical disturbance, so we also require that a_1 be non-disturbing in the EPR sense, and likewise for a_2 and a' . Therefore for simplicity in the below I assume that there is no signalling possible from α to β so that all a -type measurements are non-disturbing on β in the EPR sense. This allows Bohr’s notion of non-disturbance to be formalized as follows (where δ denotes the Kronecker δ -function)

Definition 10 (No Disturbance à la Bohr).

$$\neg \mathcal{D}\text{ist}^{\text{Bohr}}(\beta|a_1, c) \iff \forall a', A_1, \exists a_2 : [t(a_2) > t(A_1)] \wedge \left[\forall A', B, b, f(B|A', a', b, c) = \sum_{A_2, A_1} f(B|A_2, A_1, a_2, a_1, b, c) \delta_{A', A_2} \right].$$

Some comments are in order. First, just as with the EPR notion of disturbance, I have formulated Bohr's notion in an operational way, with no reference to quantum mechanics or any other theory. Second, in saying that a suitable a_2 exists for all A_1 I am allowing for an *adaptive* measurement [18], in which the choice of the second measurement a_2 may depend upon the outcome A_1 of the first measurement a_1 . Third, I have been a bit loose with notation in using $f(B|A_2, A_1, a_2, a_1, b, c)$, when my Definition 1 allowed only one measurement for each party. I trust the reader can understand the generalization implied by the new notation here. Fourth, I again stress that to avoid overly long expressions, the above is to be understood to apply only when $\neg \mathcal{D}\text{ist}^{\text{EPR}}(\beta|a, c)$ for $a \in \{a_1, a_2, a'\}$.

Bohr does not formally explain how his concept of disturbance impacts upon EPR's notion of physical reality, but it is easy to do so. The EPR argument relies upon $\neg \mathcal{D}\text{ist}(\beta|q_\alpha, c)$ being true. This is the case by EPR's concept of disturbance, but not by Bohr's, as can be seen as follows. Consider, in the EPR scenario, if Alice were to measure p_α instead of q_α . This would give the conditional probability $f(P_\beta|P_\alpha, p_\alpha, p_\beta, c) = \delta(P_\beta + P_\alpha)$. But if Alice does measure q_α then this projects Bob's state into a \hat{q}_β -eigenstate, with infinite momentum uncertainty. There is nothing Alice can do in her lab anymore to enable her to gain any information about p_β ; there is no subsequent measurement a_2 she can perform such $f(P_\beta|A_2, Q_\alpha, a_2, q_\alpha, p_\beta, c) = \delta(P_\beta + A_2)$. Thus from Definition 10, $\neg \mathcal{D}\text{ist}^{\text{Bohr}}(\beta|q_\alpha, c)$ is false. Consequently, with Bohr's notion of disturbance, \hat{q}_β is not an element of physical reality, and the EPR incompleteness proof fails.

4.2. Further commentary

Bohr's response explicitly rejects EPR's Assumption 8 that with non-interacting systems a measurement on one should not disturb the other. One could even formalize Bohr's conviction that the debate centred on an 'essential ambiguity' as a theorem:

Theorem 11 (Bohr, 1935).

$$\exists \phi \in Q : \exists a : \neg \mathcal{D}\text{ist}^{\text{EPR}}(\beta|a, c) \wedge \mathcal{D}\text{ist}^{\text{Bohr}}(\beta|a, c).$$

One could also epitomize Bohr's response in terms drawing upon (but inimical to) EPR:

Any rational⁵ definition of reality must be expected to permit that the reality of properties of the second system may depend upon the process of measurements carried out on the first system, even when it does not disturb the second system in any *mechanical* way.

⁵Here I have replaced EPR's 'reasonable' (which is perhaps best understood to mean 'locally causal' [20]) by 'rational' since Bohr states [12] that EPR's argument "discloses . . . an inadequacy of the customary viewpoint of natural philosophy for a rational account of [quantum] phenomena."

It has been argued in Ref. [27] (p. 9), that Bohr was a radical positivist, identifying reality with what can be known. My analysis of his position certainly seems compatible with that conclusion. The authors of Ref. [27] further maintain that Bohr's argument against EPR boils down to precisely that which EPR had anticipated [1],

One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted.

My analysis, by contrast, shows that the point on which he disputes with EPR is actually more subtle, which can be best appreciated as follows. As noted in the last paragraph of Sec. 3, the EPR definitions and criteria do not actually require one to consider the simultaneous existence of q_β and p_β as elements of physical reality in order to obtain a contradiction with the completeness of OQT from the EPR correlations. Deriving $\mathcal{EPR}(q^\beta)$ alone is sufficient to obtain the contradiction. Thus if the thrust of Bohr's counter-argument had been to establish that $\mathcal{EPR}(q_\beta) \wedge \mathcal{EPR}(p_\beta)$ cannot be true, it would not work against the simplified EPR argument presented at the end of Sec. 3. Luckily for Bohr, his actual argument, based on replacing EPR's Criterion 7 of disturbance with his own Definition 10, is sufficient to prevent the derivation of $\mathcal{EPR}(q_\beta)$ in itself, as noted above.

In Einstein's considered response to Bohr more than a decade later [28], he implicitly rejected Bohr's notion of disturbance in saying

One can escape from this conclusion [that orthodox quantum theory is incomplete] only by either assuming that the measurement of $[\alpha]$ (telepathically) changes the real situation of $[\beta]$ or by denying independent real situations as such to things which are spatially separated from each other. Both alternatives appear to me equally unacceptable.

Bohr's equating of a disturbance of β with a change in what Alice could *find out* about β via measurement of α , is an instance of the second of Einstein's unacceptable alternatives, namely 'denying [an] independent real situation' to β . It must be noted however that this denial by Bohr was not special to a scenario with two spatially separated systems; Bohr seems to have denied altogether that quantum things had 'real situations' in Einstein's sense. Schrödinger, by contrast, who clung to an interpretation of the wavefunction as both realistic and complete, was compelled to accept the first of Einstein's loopholes, the 'telepathic change' which Schrödinger called 'steering' [3], as a "necessary and indispensable feature" of QM. But he found this 'repugnant' and hoped that QM would be found incorrect in its prediction of steering [29], a hope which only recently has been conclusively dashed by experiment [30].

5. EPR and Entanglement

EPR's proof (using their notion of disturbance) of the incompleteness of OQT relies critically on the existence of entanglement. Indeed, for any pure entangled state a proof similar to EPR's can easily be constructed if one forgoes the pair of complementary

observables (as at the end of Sec. 3.2), and considers the observables \hat{a} and \hat{b} whose eigenstates define the Schmidt basis [31] for the entangled state. These are the observables such that a measurement of \hat{a} enables the results of a measurement of \hat{b} to be predicted (and *vice versa*). In all of the early papers in quantum foundations, the states considered are pure. Nevertheless it is also easy to prove that even allowing for mixture, an unentangled state cannot possibly be used to prove incompleteness from EPR's axioms, as I now show.

An unentangled quantum state (an operator on the tensor product of the α Hilbert space by the β Hilbert space) by definition can be written as

$$\varrho_c = \sum_k \wp_c(k) \rho_k^\alpha \otimes \rho_k^\beta, \quad (2)$$

where $\wp_c(k)$ is a probability distribution and ρ_k^α and ρ_k^β are also quantum states. In this case, a measurement on system α will update the state of system β in a purely classical way, from $\sum_k \wp_c(k) \rho_k^\beta$ to $\sum_k \wp(k|A, a) \rho_k^\beta$, where $\wp(k|A, a)$ is determined via Bayes theorem from $\wp_c(k)$ and the properties of \hat{a} and ρ_k^α .

Now if the EPR argument is to work there must be observables such that a measurement of \hat{a} enables the results of a measurement of \hat{b} to be predicted. This will be possible only if there is a decomposition of the form (2) where each ρ_k^β is an eigenoperator of \hat{b} ; that is, $\hat{b} \rho_k^\beta = B(k) \rho_k^\beta$. Then, if and only if $\rho_k^\alpha \rho_{k'}^\alpha = 0$ whenever $B(k) \neq B(k')$, there exists a measurement a that allows Alice to predict \hat{b} perfectly. In this case, one can derive, following the EPR argument, the result $\mathcal{EPR}(\hat{b}|c)$. However, unlike in the pure entangled case, there is, even for \hat{b} nontrivial, no contradiction with the assumption of completeness: there is a $\theta \in O_\phi$ in which \hat{b} is represented. It is the version of OQT which interprets the index k in Eq. (2) as a variable in the theory, a variable like λ in the Definition 2, with $\int_{\Lambda_\theta} d\mu_c^\theta(\lambda) \rightarrow \sum_k \wp_c(k)$. This version of OQT is the one which takes the appropriate mixture (2) to be a *proper* mixture [32]. Then, from the Definition 4, $\mathcal{Rep}_\theta(\hat{b}|c)$ is true, since \hat{b} has a definite value given k , namely $B(k)$.

Of course there are other OQTs, in particular the one in which there are no variables in the theory apart from the ϱ_c which is uniquely specified by the preparation c , so that all decompositions (2) are regarded as *improper* mixtures [32]. This particular $\theta \in O_\phi$ is provably incomplete according to EPR's definition of disturbance. Moreover, adopting Bohr's definition of disturbance does *not* necessarily save the day in this case. That is, there are cases where Bohr would also be forced to say that there is no disturbance, as I will discuss at the end of the following section. Thus, unless one believes that Bohr would have allowed that OQT is incomplete, one must accept that the question of incompleteness of OQT should be answered negatively only if *every* $\theta \in O_\phi$ is incomplete⁶. This incompleteness for all versions OQT is of course what EPR did prove (again according to their definition of disturbance), but what we have seen here is that entanglement is necessary (and, if pure, sufficient) for the EPR Theorem 9.

⁶The situation is similar to that in classical physics. There are some classical physical theories, such as thermodynamics, that are incomplete in that they do not postulate any microscopic variables. But that does not mean that classical physics *per se* is incomplete.

6. Bohr and Discord

Given the discussion in the preceding section on the criticality of entanglement for the EPR theorem, it might be expected that entanglement is equally critical for Theorem 11 (which I have called Bohr's theorem), that there are quantum phenomena in which even though mechanical disturbance is excluded, a measurement on α can still Bohr-disturb system β . Surprisingly, this is not the case. Rather, it turns out that what is critical to Bohr's theorem is the existence of *quantum discord*. Note that in this section I am concerned with Bohr's non-mechanical disturbance in itself, not its role in his counter-argument to EPR, which has already been treated in Sec. 4. I will return to the EPR-Bohr debate in Sec. 7.

As explained briefly in the introduction, discord was a term introduced by Ollivier and Zurek [8] in 2002 as a quantitative property of a quantum state ϱ_c shared between two parties (say Alice and Bob), only one of which (say Alice) may perform a measurement a . They defined Alice-discord as

$$\delta(\beta|a, c) = I(\beta : \alpha|c) - J(\beta|a, c), \quad (3)$$

where I is the quantum mutual information between Alice's quantum system α and Bob's β , and J is the average information Alice has about β from the result of her measurement a (and her knowledge that the original shared state was ϱ_c). In classical information theory Alice's measurement could simply reveal the configuration of her system in which case the difference δ between would be zero by definition. In quantum information theory this is not the case, and for some states ϱ_c , the Alice-discord $\delta(\beta|a, c)$ is nonzero for all possible measurements. Henderson and Vedral had previously introduced this quantity without naming it, and considered its minimum over all possible measurements, which I will denote $\delta(\beta|c)$.

The proofs below do not use the magnitude of discord, only whether it is nonzero, which I notate by

Definition 12 (Discord).

$$\text{Disc}(\beta|c) \iff \delta(\beta|c) > 0.$$

Now it is well known that $\text{Disc}(\beta|c)$ is false if and only if

$$\varrho_c = \sum_j \wp_c(j) \pi_j^\alpha \otimes \rho_j^\beta, \quad (4)$$

where $\wp_c(j)$ is a probability distribution, the π_j^α s are mutually orthogonal rank-one projectors, and the ρ_j^β s are simply normalized states with no other restrictions. This is of course an unentangled state, so from Sec. 5 any measurement by Alice on such a state would result in classical updating of her knowledge of Bob's state. We now show how Alice-discord is intimately related to Bohr's notion of disturbance, in that

$$\forall \phi \in Q, \quad [\neg \text{Disc}(\beta|c) \iff \forall \hat{a}, \exists a \in \mathbb{S}_{\hat{a}|c} : \neg \text{Dist}^{\text{Bohr}}(\beta|a, c)]. \quad (5)$$

To prove the forward implication, assume that $\text{Disc}(\beta|c)$ is false. Then one way in which Alice can measure an observable \hat{a}_1 is as follows. First she measures the observable

\hat{a}_c for which π_j^α are eigenoperators. This would update her knowledge of Bob's state to ρ_j^β for some j , and, if done by projection, would leave her system in the corresponding state π_j^α . On average, (that is, ignoring the result A_c which stores the value j) this measurement has no effect on the state. Thus Alice can then measure \hat{a}_1 by any other means as if she had never measured \hat{a}_c , and update her knowledge of β thereby. But then recalling the result A_c she can restore the initial state of the system by re-preparing her system in state π_j^α again, and then again forgetting A_c . Thus with the trivial choice $a_2 = a'$, a subsequent measurement of a_2 will yield the same information as if the measurement of a' had been performed instead of a_1 . In other words, by the Definition 10, this measurement of a_1 is non-disturbing.

Now to prove the reverse implication, assume instead that all observables \hat{a} can be measured without Bohr-disturbing the system β . For a quantum system, this means that any \hat{a} can be measured in a way that (once the result is forgotten) the joint system is left in the same state ϱ_c . This is only possible if the algebra of observables on α is effectively classical, with no non-commuting structure. The state (4) is simply the way to write this classicality in quantum terms. Thus it must be the case that the joint state is Alice-concordant, with $\delta(\beta|c) = 0$. Rewriting (5) in positive terms, we can thus state

Theorem 13.

$$\forall \phi \in Q, [\text{Disc}(\beta|c) \iff \exists \hat{a} : \forall a \in \mathbb{S}_{\hat{a}|c}, \text{Dist}^{\text{Bohr}}(\beta|a, c)]$$

That is, if and only if the joint state is Alice-discordant (with $\delta(\beta|c) > 0$) then there is some α -observable such that the measurement thereof necessarily disturbs Bob's system β .

The discord $\delta(\beta|a, c)$ as defined in Eq. (3) can be thought of as the minimum amount of quantum information about β that is irrevocably lost when Alice performs a measurement a . The loss occurs precisely because by choosing a particular measurement, Alice is eliminating “possible types of predictions regarding the future behaviour of the system”. The discord $\delta(\beta|c)$ is in fact a *quantification*, in bits, of Bohr's non-mechanical disturbance, the extent to which the measurement a influences “the very conditions which define the possible types of predictions”.

It is the *complementarity* of different observables, the fact that they “demand mutually exclusive experimental arrangements” [12], that is the essence of the phenomenon of discord, and makes Bohr's notion of non-mechanical disturbance non-trivial. The trivial aspect to Bohr's notion is that any measurement a can be disturbing, according to the Definition 10: simply by destroying the system α Alice will lose information about the distant system β . The non-trivial point is that, for a suitable state, complementarity can make this disturbance inevitable for at least one observable \hat{a} . In fact, it is a plausible conjecture that disturbance is inevitable for *all* observables \hat{a} with non-degenerate eigenvalues.

The role of complementarity in classifying bipartite quantum states has previously been discussed in Refs. [33, 34]. However both of these papers considered the question of whether complementarity could play any role in the system at all, and concluded that it did not only when the system was of the form

$$\varrho_c = \sum_j \wp_c(j) \pi_j^\alpha \otimes \pi_j^\beta, \quad (6)$$

where the π_j^β are also mutually orthogonal rank-one projectors. These are simply states that are both Alice-concordant and Bob-concordant and may be called consonant states (in that they have zero *dissonance* [35]). While these conclusions are certainly valid, in the context of understanding Bohr's notion of disturbance it is more enlightening to concentrate upon the asymmetric notion of discord, just as for understanding the EPR phenomenon it is useful to introduce the asymmetric notion of steering. The reason is the same: both EPR and Bohr were interested in the consequences, for one system (Bob's), of making a measurement upon a distant second system (Alice's).

Although not central to the discussion, it is worth highlighting the hierarchy of the nonlocality concepts relating to bipartite states which have been mentioned in this paper:

$$\text{Bell-nonlocality} \implies \text{steering} \implies \text{entanglement} \implies \text{discord} \implies \text{dissonance}$$

Note that for the asymmetric concepts (steering and discord), the implications apply regardless of which party we specify it for. Note also that all of these implications are strict (the converse implications are false). Finally, returning to the point at the end of the preceding section, it is for the case of (non-product) consonant states that there exists a (non-trivial) element of physical reality \hat{b} by the EPR Criterion 5, whether by EPR's or Bohr's notion of disturbance. This is because such states have perfect correlations in the bases used in Eq. (6) and yet, being concordant, there is no disturbance à la Bohr.

7. Discussion

I have suggested that both the EPR argument, and Bohr's counter-argument, can be better appreciated by attempting a rigorous formulation. In particular, my formalization allows us to see that both were correct, and that their differing conclusions about the completeness of quantum mechanics hinged upon a differing conception of disturbance. For EPR, as long as two quantum systems no longer interact, a measurement performed by Alice's has no measurable effect on Bob's potential outcomes, and so does not disturb Bob's system. For Bohr, if Alice's measurement alters the possible predictions she could otherwise have made regarding Bob's potential outcomes then that amounts to a disturbance of Bob's system. I have shown that Bohr's concept is intimately related to the concept of quantum discord, in that if and only if the state they share has no Alice-discord, there is a way for her to measure any observable in a way that entails no disturbance of Bob's system, in Bohr's sense.

As originally introduced, discord was a quantitative measure of the necessary loss of quantum information about Bob's system when Alice performs a measurement. Such a quantification is only possible when an information theory exists. This is the case for classical probabilities and for quantum states and measurements. For more general theories, such as Spekkens' toy-bit theory [36], and the PR-box theory [37], this is not necessarily the case. In this context, the present work shows that it is still possible to consider the presence or absence of discord by using the operational notion of Bohr-disturbance. In particular, in toy-bit theory, at least for the case of a few toy-bits each held by Alice and Bob, it is not possible to exhibit disturbance in Bohr's sense for LOCC-preparable bipartite states. That is, states that can be prepared from uncorrelated states by local operations and classical communication (here via a toy-bit channel which decoheres in a particular basis, so to speak) between Alice and Bob.

Returning now to quantum mechanics, the situation is unlike that of the toy theory, as there definitely exist non-entangled (and thus LOCC-preparable) states that are discordant. As I have argued in Sec. 5, entanglement is a necessary property, and pure entanglement a sufficient property, of quantum states for a proof of incompleteness using EPR’s definitions. By contrast, discord is the necessary and sufficient phenomenon that underpins Bohr’s counter-argument. Thus there is no evidence in Bohr’s reply that he engaged with the EPR phenomenon of steering, nor with the concept of entanglement which it relies upon.

EPR’s result (Theorem 9) states that orthodox quantum theory is incomplete, by EPR’s necessary criterion for completeness that all elements of physical reality are represented in the theory. Now the existence of elements of physical reality is an operational concept for EPR, and for maximally entangled states [31] all properties of β would be elements of physical reality by this definition, because Alice can predict any property of β by measuring her system in the appropriate basis. Likewise all properties of α would be elements of physical reality. Thus, in a complete theory, these should be represented, by having values determined by the theory, independent of measurements made on the other system — see Sec. 3.1. The impossibility of such a local representation, in *any* theory, for the correlations of maximally entangled qubit states, was exactly what Bell proved in 1964 [6]. That is, Bell’s 1964 theorem can be stated, in precisely EPR’s terms, as:

Theorem 14 (Bell, 1964).

$$\exists \phi \in Q : \forall \theta \in \Theta_\phi, \neg \text{Com}^{\text{EPR}}(\theta).$$

Here I use the EPR superscript on $\text{Com}^{\text{EPR}}(\theta)$ to emphasize that this is EPR’s notion of completeness, and in particular uses EPR’s criterion for disturbance, not Bohr’s.

If Bell’s theorem had been intuited by Bohr then it would of course have been the perfect response to the EPR paper: “Your criterion for completeness implies that no theory for quantum phenomena can provide a complete description. So you must admit that either your concept of completeness is misguided, or that your conviction that ‘such a theory is possible’ [1] is mistaken.” The present work militates against the plausibility of an alternate history such as this (or that in Ref. [21]). The structure of Bohr’s response to EPR is in no way anticipatory of Bell-nonlocality. Rather, Bohr’s argument relied upon the phenomenon of quantum discord, a notion of quantum correlations with no requirement for entanglement, or steering, let alone Bell-nonlocality. It is no coincidence that Schrödinger, who coined the terms entanglement and steering, and Bell, whose theorem overshadows all of the 1935 papers, were followers of Einstein, not Bohr.

Acknowledgements

I am grateful to Eric Cavalcanti for long- and far-ranging discussions, to Terry Rudolph for stimulating remarks, and to Eric Cavalcanti, Rob Spekkens and Travis Norsen for detailed criticisms of the manuscript.

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